A DIOPHANTINE EQUATION OF ANTONIADIS

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ABSTRACT. It is proved that the diophantine equation $31 \cdot x^2 + 1 = y^3$ has exactly the solutions $(x,y) = (0,1), (2,5), (-2,5)$, by solving a related Thue-equation, using the theory of linear forms in logarithms and computational methods from diophantine approximation theory.

1. INTRODUCTION

In 1983 Jannis A. Antoniadis conjectured, in connection with a study of imaginary quadratic fields with class number 2 and diophantine equations, that the diophantine equation

$$31 \cdot x^2 + 1 = y^3 \tag{1}$$

has only the solutions $(x,y) = (0,1), (2,5), (-2,5)$ (cf. Antoniadis [1983], Section 2.5). It is the aim of this paper to prove this conjecture.

The proof consists of three main steps. The first step is to reduce equation (1) to a couple of Thue-equations, of which one is trivially solvable, and of which the other is

$$X^3 - 6 \cdot X \cdot Y^2 + Y^3 = 62 \ . \tag{2}$$

The second step is to derive from the theory of linear forms in logarithms an upper bound for the solutions of (2). This we do following Tzanakis and de Weger [1987]. The third step is to reduce this upper bound to a level that makes it possible to enumerate the possibilities below this bound. We use a computational method from diophantine approximation theory, based on a continued fraction algorithm, as a two-dimensional analogue of the multi-dimensional L^3-lattice basis reduction algorithm, following Tzanakis and de Weger [1987] (see also de Weger [1987]). Thus we prove that (2) has only one solution: $(X,Y) = (-7,3)$.

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2. THE REDUCTION TO THUE-EQUATIONS

Let $K = \mathbb{Q}(\sqrt{-31})$. It has class number 3 , and its ring of integers is $\mathcal{O}_K = \mathbb{Z} + \frac{1+\sqrt{-31}}{2}\cdot\mathbb{Z}$. The prime 2 splits in $\mathcal{O}_K$ as $(2) = \mathfrak{p}\cdot\bar{\mathfrak{p}}$, say, with $\mathfrak{p}^3 = \left(\frac{1+\sqrt{-31}}{2}\right)$. Let x, y be a solution of (1), and put $\mathfrak{b} = (1+x\cdot\sqrt{-31})$. Equation (1) leads to an equation of ideals in $\mathcal{O}_K$:

$$(y)^3 = \mathfrak{b}\cdot\bar{\mathfrak{b}} \ . \tag{3}$$

A common divisor of $\mathfrak{b}$ and $\bar{\mathfrak{b}}$ must divide $\mathfrak{b} + \bar{\mathfrak{b}} = (2)$, and it must be its own conjugate. Hence it can only be (1) or (2) . Suppose that it is (2) . Since $\mathfrak{b}$ is a principal ideal, there exists a $\xi \in \mathcal{O}_K$ such that $\mathfrak{b} = (2)\cdot(\xi)$, and $\bar{\mathfrak{b}} = (2)\cdot(\bar{\xi})$, such that (ξ) and $(\bar{\xi})$ are coprime. It follows that $y = 2\cdot y_1$ for a $y_1 \in \mathbb{Z}$, and we find $(\xi)\cdot(\bar{\xi}) = \mathfrak{p}\cdot\bar{\mathfrak{p}}\cdot(y_1)^3$. Because (ξ) and $(\bar{\xi})$ are coprime, we have $(\xi) = \mathfrak{p}\cdot\mathfrak{a}^3$ or $(\xi) = \bar{\mathfrak{p}}\cdot\mathfrak{a}^3$ for some ideal $\mathfrak{a}$. This however is impossible, since $\mathfrak{a}^3$ and (ξ) are principal ideals, whereas $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ are not.

We conclude that $\mathfrak{b}$ and $\bar{\mathfrak{b}}$ are coprime, and from (3) it then follows that they are cubes, $\mathfrak{b} = \mathfrak{a}^3$ and $\bar{\mathfrak{b}} = \bar{\mathfrak{a}}^3$, say. Representations of the three ideal classes of $\mathcal{O}_K$ are (1) , $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ respectively. So there exists an $\alpha \in K$ such that $\mathfrak{a} = (\alpha)$, or $\mathfrak{a} = \mathfrak{p}\cdot(\alpha)$, or $\mathfrak{a} = \bar{\mathfrak{p}}\cdot(\alpha)$, respectively. The last two cases are equivalent, on taking conjugates (note that the sign of x is not important). So we consider the first two cases only.

Suppose first that $\mathfrak{a} = (\alpha)$. Then $\alpha \in \mathcal{O}_K$, and from $\mathfrak{b} = (\alpha^3)$ it follows that $1 + x\cdot\sqrt{-31} = \pm\alpha^3$. Without loss of generality we may neglect the $\pm$-sign. Put $\alpha = \frac{u+v\cdot\sqrt{-31}}{2}$ for $u, v \in \mathbb{Z}$. Then

$$1 + x\cdot\sqrt{-31} = \frac{u^3 - 93\cdot u\cdot v^2}{8} + \frac{3\cdot u^2\cdot v - 31\cdot v^3}{8}\cdot\sqrt{-31} \ ,$$

and equating the real parts yields the Thue-equation

$$u^3 - 93\cdot u\cdot v^2 = 8 \ .$$

It follows that $u \mid 8$, and by checking the few possibilities, we find that only $(u,v) = (2,0)$ is a solution. It leads to the solution $(x,y) = (0,1)$ for (1).

Next suppose $\mathfrak{a} = \mathfrak{p}\cdot(\alpha)$. Then $\bar{\mathfrak{p}}\cdot\mathfrak{a} = (2\cdot\alpha)$, hence $2\cdot\alpha \in \mathcal{O}_K$. Put $\alpha = \frac{u+v\cdot\sqrt{-31}}{4}$ for $u, v \in \mathbb{Z}$. Then from $\mathfrak{b} = \mathfrak{a}^3 = \mathfrak{p}^3\cdot(\alpha^3)$ we find

$$1 + x\cdot\sqrt{-31} = \pm\left(\frac{1+\sqrt{-31}}{2}\right)\cdot\left(\frac{u+v\cdot\sqrt{-31}}{4}\right)^3 \ .$$

Without loss of generality we may neglect the ±-sign. Equating real and imaginary parts we obtain

$$u^3 - 93\cdot u^2\cdot v - 93\cdot u\cdot v^2 + 961\cdot v^3 = 128 \ , \tag{4}$$

$$u^3 + 3\cdot u^2\cdot v - 93\cdot u\cdot v^2 - 31\cdot v^3 = 128\cdot x \ . \tag{5}$$

The first one, equation (4), is a Thue-equation. Put

$$2\cdot Z = 31\cdot v - u \ , \quad Y = -u \ .$$

Note that it follows from (4) that u and v have equal parity, so that $Z \in \mathbb{Z}$. Now (4) leads to

$$Z^3 - 24\cdot Z\cdot Y^2 + 8\cdot Y^3 = 496 \ .$$

It follows that Z is even. Put $Z = 2\cdot X$, $X \in \mathbb{Z}$. Then we find the Thue- equation (2). In the next sections we show that (2) has only the solution $X = -7$, $Y = 3$. This leads to $Z = -14$, $u = -3$, $v = -1$. Then (5) yields $x = 2$, and we find $y = 5$ from (1). Of course, taking conjugates in the above reasoning, we find the solution $x = -2$, $y = 5$ for (1).

3. AN UPPER BOUND FOR THE SOLUTIONS OF THE THUE-EQUATION

We study equation (2). Let ϑ be a root of the equation

$$\vartheta^3 - 6\cdot\vartheta + 1 = 0 \ .$$

In an Appendix to this paper we give numerical approximations to the conjugates $\vartheta^{(i)}$ for $i = 1, 2, 3$.

Put $L = \mathbb{Q}(\vartheta)$. Its discriminant is $837 = 3^3\cdot 31$, and its class number is 1 (cf. Borewicz and Šafarevič [1966], Tab. 7, p. 461). Its ring of integers $\mathcal{O}_L$ is generated by 1, ϑ, ϑ^2 , and a system of fundamental units is

$$\epsilon = -\vartheta \ , \quad \eta = -4 + 6\cdot\vartheta + 3\cdot\vartheta^2$$

(cf. Pohst, Weiler and Zassenhaus [1982], Tab. I.1, p.300, who give 1, ϑ, $-4+\vartheta^2$ as generators for $\mathcal{O}_L$, and

$$-\epsilon = \vartheta \ , \quad \eta\cdot\vartheta = 21 + 14\cdot\vartheta + 6\cdot(-4+\vartheta^2)$$

as a system of fundamental units, which data are equivalent to ours). Note that $N(\epsilon) = N(\eta) = 1$.

The ideal (2) factorizes in $\mathcal{O}_L$ into prime ideals as

$$(2) = (-4 + \vartheta + \vartheta^2) \cdot (-1 + 3\cdot\vartheta - \vartheta^2) ,$$

with $N(-4+\vartheta+\vartheta^2) = 2$ and $N(-1+3\cdot\vartheta-\vartheta^2) = 4$. The ideal (31) has the prime decomposition

$$(31) = (-1 - 2\cdot\vartheta) \cdot (-1 + 4\cdot\vartheta)^2 ,$$

with $N(-1-2\cdot\vartheta) = N(-1+4\cdot\vartheta) = 31$. (We leave it as an exercise to the reader to check these prime decompositions of (2) and (31) in $\mathcal{O}_L$). It follows that there are exactly two ideals of norm 62 , namely (ψ_1) and (ψ_2) , where we choose

$$\psi_1 = (-4 + \vartheta + \vartheta^2) \cdot (-1 - 2\cdot\vartheta) = 6 - 5\cdot\vartheta - 3\cdot\vartheta^2 ,$$

$$\psi_2 = - (-4 + \vartheta + \vartheta^2) \cdot (-1 + 4\cdot\vartheta) \cdot \vartheta^{-1} = -7 - 3\cdot\vartheta .$$

Note that $N(\psi_1) = N(\psi_2) = 62$.

Let $X, Y \in \mathbb{Z}$ be a solution of the Thue-equation (2). Put

$$\beta = X - Y\cdot\vartheta .$$

Then equation (2) leads to the equation $N(\beta) = 62$ for $\beta \in \mathcal{O}_L$. The only elements in $\mathcal{O}_L$ with norm 62 are

$$\epsilon^a \cdot \eta^b \cdot \psi_\ell ,$$

where $a, b \in \mathbb{Z}$ and $\ell \in \{ 1, 2 \}$. Thus for some ℓ we have the equations

$$X - Y\cdot\vartheta^{(i)} = \epsilon^{(i)^a} \cdot \eta^{(i)^b} \cdot \psi_\ell^{(i)} \quad \text{for} \quad i = 1, 2, 3 .$$

Eliminating X and Y from these three equations we find

$$\delta_{\ell,i} \cdot \left[\frac{\epsilon^{(j)}}{\epsilon^{(k)}}\right]^a \cdot \left[\frac{\eta^{(j)}}{\eta^{(k)}}\right]^b - 1 = \frac{-1}{\delta_{\ell,j}} \cdot \left[\frac{\epsilon^{(i)}}{\epsilon^{(k)}}\right]^a \cdot \left[\frac{\eta^{(i)}}{\eta^{(k)}}\right]^b , \tag{6}$$

where we choose $\{i,j,k\}$ to be the even permutation of $\{1,2,3\}$ for which $|\beta^{(i)}|$ is minimal, and

$$\delta_{\ell,i} = \frac{\psi_\ell^{(j)}}{\psi_\ell^{(k)}} \cdot \frac{\vartheta^{(i)} - \vartheta^{(k)}}{\vartheta^{(i)} - \vartheta^{(j)}} , \quad \delta_{\ell,j} = \frac{\psi_\ell^{(k)}}{\psi_\ell^{(i)}} \cdot \frac{\vartheta^{(j)} - \vartheta^{(i)}}{\vartheta^{(j)} - \vartheta^{(k)}} .$$

Note that in equation (6) the only variables are a and b . Put

$$\mu_{1,i} = \frac{\epsilon^{(j)}}{\epsilon^{(k)}} , \quad \mu_{2,i} = \frac{\eta^{(j)}}{\eta^{(k)}} .$$

We study the linear forms in logarithms of algebraic numbers given b

$$\Lambda_{\ell,i} = \log|\delta_{\ell,i}| + a\cdot\log|\mu_{1,i}| + b\cdot\log|\mu_{2,i}|$$

for $\ell = 1, 2$ and $i = 1, 2, 3$. From (6) and the fact that i is taken such that $|\beta^{(i)}|$ is minimal, it follows that $|\Lambda_{\ell,i}|$ is small. However, we do not know a priori the values of ℓ and i , so we have to check the six possibilities. (Note that $\delta_{\ell,1}\cdot\delta_{\ell,2}\cdot\delta_{\ell,3} = -1$ and $\mu_{h,1}\cdot\mu_{h,2}\cdot\mu_{h,3} = 1$, hence $\Lambda_{\ell,1} + \Lambda_{\ell,2} + \Lambda_{\ell,3} = 0$, but this information is not of any use). We need to be more precise about how small $|\Lambda_{\ell,i}|$ is. We follow the line of argument from Tzanakis and de Weger [1987] (referred to as "*T–dW*" in the sequel), Sections II.1 and II.2, which is equivalent to de Weger [1987] (referred to as *deW* in the sequel), Ch. 8, Sections 2 and 3, and we compute the constants Y_0, Y_1, ..., C_1, C_2, ... that occur there.

Put $Y_0 = 1$, $C_1 = 4\cdot62/|3\cdot\vartheta^{(2)2}-6| < 41.93$, $C_2 = (\vartheta^{(3)}-\vartheta^{(2)})/2 > 1.097$, $Y_1 = 168$. Then *T–dW*, Lemma 1.1 or *deW*, Lemma 8.1 yields:

$$\text{if } |Y| \ge 2 \text{ then } |\beta^{(i)}| < 41.93\cdot|Y|^{-2} ,$$

$$\text{and } \min\{\,|\beta^{(j)}|, |\beta^{(k)}|\,\} > 1.097\cdot|Y| .$$

Put $C_3 = |\vartheta^{(3)}-\vartheta^{(1)}|/|\vartheta^{(3)}-\vartheta^{(2)}| < 2.229$, $Y_2 = 168$. Then *T–dW*, Lemma 1.2 or *deW*, Lemma 8.2 yields

$$\text{if } |Y| \ge 169 \text{ then } |\Lambda_{\ell,i}| < 118.5\cdot|Y|^{-3} .$$

Put $A = \max\{\,|a|, |b|\,\}$. The minimal $N[U_I^{-1}]$, defined as in *T–dW*, Lemma 2.1 or *deW*, Lemma 8.3, is less than 0.6104 , and the maximal $N[U_I^{-1}]$ is less than 0.7776 . We have $\mu_- = |\psi_1^{(1)}| > 0.5416$, $\mu_+ = |\psi_1^{(3)}| < 22.537$, $C_4 < 9.953$, $C_5 < 0.7776$. Hence, from *T–dW*, Lemma 2.1 or *deW*, Lemma 8.3 it follows that

$$\text{if } |Y| \ge 169 \text{ then } A < 0.7776\cdot\log(9.953\cdot|Y|) .$$

T–dW, Lemma 2.2 or *deW*, Lemma 8.4 then yields, by $C_6 < 1.1677\times10^5$ and $3/C_5 > 3.858$, that

$$\text{if } |Y| \geq 169 \text{ then } |\Lambda_{\ell,i}| < 1.1677\times 10^5 \cdot e^{-3.858\cdot A} . \qquad (7)$$

The solutions of (2) with $|Y| \leq 168$ are easy to find by a small computation. There is only one: $X = -7$, $Y = 3$. From now on we therefore assume that $|Y| \geq 169$.

The next step is to apply the theory of linear forms in logarithms, to obtain a lower bound for $|\Lambda_{\ell,i}|$, as in Waldschmidt [1980], cf. also *T–dW*, Appendix II, and *deW*, Lemma 2.4. Consider the smallest algebraic number field M that contains all conjugates of ϑ, namely $M = \mathbb{Q}(\vartheta^{(1)},\vartheta^{(2)})$. It has degree 6. The algebraic numbers $\delta_{\ell,i}$ and $\mu_{h,i}$ are elements of this field. We are looking for the leading coefficients of their defining polynomials. Some laborious computations yield as defining polynomial for $\delta_{1,i}$:

$$119164\cdot(x^6+1)+8931534\cdot(x^5+x)+152559285\cdot(x^4+x^2)+382415705\cdot x^3 ,$$

with leading coefficient $119164 = 4\cdot 31^3$, and for $\delta_{2,i}$:

$$4\cdot(x^6+1) - 12\cdot(x^5+x) - 3351\cdot(x^4+x^2) + 6722\cdot x^3 ,$$

with leading coefficient 4. This information enables us to compute the absolute logarithmic heights of the $\delta_{\ell,i}$. We have, by the fact that $\delta_{\ell,1}$, $\delta_{\ell,2}$, $\delta_{\ell,3}$, $\delta_{\ell,1}^{-1}$, $\delta_{\ell,2}^{-1}$, $\delta_{\ell,3}^{-1}$ are conjugates, the following expression for this height, according to Waldschmidt [1980]:

$$h(\delta_{\ell,i}) = \frac{1}{6}\cdot\log\left(a_0\cdot\prod_{j=1}^{3}\max\left(|\delta_{\ell,j}|,|\delta_{\ell,j}^{-1}|\right)\right) \quad \text{for } i = 1, 2, 3 .$$

We give the following approximations of the $\delta_{\ell,i}$:

$$\begin{aligned} \delta_{1,1} &= -0.40871087\ldots , & \delta_{2,1} &= 0.96610162\ldots , \\ \delta_{1,2} &= -51.13098544\ldots , & \delta_{2,2} &= 29.49993845\ldots , \\ \delta_{1,3} &= -0.04785195\ldots , & \delta_{2,3} &= -0.03508779\ldots . \end{aligned}$$

Hence for $i = 1, 2, 3$ we find

$$h(\delta_{1,i}) = \frac{1}{6}\cdot\log\left(119164\cdot|\delta_{1,1}^{-1}|\cdot|\delta_{1,2}|\cdot|\delta_{1,3}^{-1}|\right) < 3.2596 ,$$

$$h(\delta_{2,i}) = \frac{1}{6}\cdot\log\left(4\cdot|\delta_{2,1}^{-1}|\cdot|\delta_{2,2}|\cdot|\delta_{2,3}^{-1}|\right) < 1.3592 .$$

We also want to compute the heights of the $\mu_{h,i}$. The leading coefficients of their defining polynimials are equal to 1, since they are quotients of units, hence algebraic integers. The conjugates are the

μ's and their inverses. We have approximately:

$$\mu_{1,1} = 0.07090891\ldots\ , \quad \mu_{2,1} = -0.10822894\ldots\ ,$$
$$\mu_{1,2} = -0.93378623\ldots\ , \quad \mu_{2,2} = 2106.31618305\ldots\ ,$$
$$\mu_{1,3} = -15.10259882\ldots\ , \quad \mu_{2,3} = 0.00438665\ldots\ .$$

Hence for $i = 1, 2, 3$

$$h(\mu_{1,i}) = \frac{1}{6}\cdot\log\left(|\mu_{1,1}^{-1}|\cdot|\mu_{1,2}^{-1}|\cdot|\mu_{1,3}|\right) < 0.9050\ ,$$

$$h(\mu_{2,i}) = \frac{1}{6}\cdot\log\left(|\mu_{2,1}^{-1}|\cdot|\mu_{2,2}|\cdot|\mu_{2,3}^{-1}|\right) < 2.5509\ .$$

In Waldschmidt's bound we now take

$$\max\left(h(\mu_{1,i}),\log|\mu_{1,i}|/6\right) < 0.9050 = V_1\ ,$$

$$\max\left(h(\mu_{2,i}),\log|\mu_{2,i}|/6\right) < 2.5509 = V_2 = V_2^+\ ,$$

$$\max\left(h(\delta_{\ell,i}),\log|\delta_{\ell,i}|/6\right) < 3.2596 = V_3 = V_3^+\ .$$

We have further $n = 3$, hence $e(n) = 63$, and $D = 6$. So if $\Lambda_{\ell,i} \neq 0$ then Waldschmidt [1980] (cf. *T-dW*, Lemma 2.3) yields

$$|\Lambda_{\ell,i}| > \exp\left(-C_7\cdot(\log A + C_8)\right)\ , \tag{8}$$

where

$$C_7 = 2^{63}\cdot 3^6\cdot 6^5\cdot 0.9050\cdot 2.5509\cdot 3.2596\cdot\log(e\cdot 6\cdot 2.5509) < 1.4669\times 10^{27}\ ,$$

$$C_8 = \log(e\cdot 6\cdot 3.2596) < 3.9734\ .$$

Since the right hand side of (6) cannot be equal to 0, it follows that $\Lambda_{\ell,i} \neq 0$. Combining (7) and (8) we find:

$$A < 2.64\times 10^{28}\ .$$

4. REDUCING THE UPPER BOUND

In this section we perform the computations that make it possible to reduce considerably the upper bound 2.64×10^{28} for $A = \max(|a|,|b|)$ for the solutions of

$$|\Lambda_{\ell,i}| < 1.1677\times 10^5\cdot e^{-3.858\cdot A}\ , \tag{9}$$

where

$$\Lambda_{\ell,i} = \log|\delta_{\ell,i}| + a\cdot\log|\mu_{1,i}| + b\cdot\log|\mu_{2,i}| \ ,$$

for the six cases $\ell = 1, 2$, $i = 1, 2, 3$.

The method that we use is that of *T-dW*, Section II.3 (see also *deW*, Sections 3.8, 8.4). The linear forms have homogeneous parts with two terms only, so that the approximation lattices will have dimension 2. We also could have applied the well known continued fraction method of Baker and Davenport [1969] (see also *deW*, Section 3.3). However, we want to show that the alternative method works as well in the one-dimensional case.

In this case it is not necessary to apply the L^3-algorithm to find reduced bases of the lattices, due to the fact that the lattices are only two- dimensional. It is sufficient to apply some continued fraction algorithm, with which one is able to find explicitly the shortest nonzero vector in the lattice. This vector occurs as the first vector of the reduced basis, whereas the second vector of the reduced basis is the shortest lattice vector independent of the first one.

For $i = 1, 2, 3$ we define the lattices Γ_i as spanned by the column vectors of the matrices

$$\mathcal{B}_i = \begin{pmatrix} 1 & 0 \\ [C_0\cdot\log|\mu_{1,i}|] & [C_0\cdot\log|\mu_{2,i}|] \end{pmatrix} .$$

Here, $[x]$ means $\lfloor x \rfloor$ if $x \geq 0$, and $\lceil x \rceil$ if $x < 0$, in other words, rounded off towards zero. Note that the lattices Γ_i depend on i only, not on ℓ . Further, put

$$\mathcal{Y}_{\ell,i} = \begin{pmatrix} 0 \\ -[C_0\cdot\log|\delta_{\ell,i}|] \end{pmatrix} .$$

The constant C_0 we take somewhat larger than the square of the upper bound for A , as the heuristics prescribe (cf. *deW*, Section 3.8). In our case, we took initially $C_0 = 10^{64}$, which is somewhat larger than $(2.64\times10^{28})^2$.

We computed the values of $\log|\mu_{h,i}|$ and $\log|\delta_{\ell,i}|$ for $h = 1, 2$, $i = 1, 2, 3$, $\ell = 1, 2$ to 65 decimal places behind the decimal point, so that the lattices $\mathcal{B}_i$ are known explicitly. We give the numerical values in the Appendix. We computed reduced bases for these lattices, which also are given explicitly in the Appendix. It follows that the length of the shortest nonzero lattice vector $\underline{b}_{1,i}$ in all three cases satisfies

$$|b_{1,i}| > 7.904\times10^{31} .$$

Let $\underline{s}_{\ell,i} \in \mathbb{Q}^2$ be such that

$$\mathcal{B}_i \cdot \underline{s}_{\ell,i} = \underline{y}_{\ell,i} .$$

We computed all $\underline{s}_{\ell,i}$, and give their values in the Appendix. It follows that for all their coordinates the distance to the nearest integer is at least 0.02147. Now we apply *dcW*, Lemma 3.5, which yields

$$\min_{\underline{x}\in\Gamma_i} |\underline{x}-\underline{y}_{\ell,i}| > 2^{-1/2}\cdot 0.02147\cdot 7.904\times10^{31} > 1.199\times10^{30} . \qquad (10)$$

We now take the lattice point

$$\underline{x} = \mathcal{B}_i \cdot \begin{pmatrix} a \\ b \end{pmatrix} ,$$

where a, b are the coefficients of the linear form $\Lambda_{\ell,i}$. Put

$$\lambda_{\ell,i} = [C_0\cdot\log|\delta_{\ell,i}|] + a\cdot[C_0\cdot\log|\mu_{1,i}|] + b\cdot[C_0\cdot\log|\mu_{2,i}|] .$$

Then

$$\underline{x} - \underline{y}_{\ell,i} = \begin{pmatrix} a \\ \lambda_{\ell,i} \end{pmatrix} ,$$

hence (10) yields

$$a^2 + \lambda_{\ell,i}^2 > (1.199\times10^{30})^2 .$$

By $|a| \le A < 2.64\times10^{28}$ we obtain

$$|\lambda_{\ell,i}| > 1.198\times10^{30} .$$

On the other hand, by the definitions of $\Lambda_{\ell,i}$ and $\lambda_{\ell,i}$,

$$|\lambda_{\ell,i} - 10^{64}\cdot\Lambda_{\ell,i}| \le 1 + |a| + |b| \le 1 + 2\cdot A < 5.29\times10^{28} .$$

Hence

$$|\Lambda_{\ell,i}| > 1.145\times10^{-34} .$$

Then (9) yields $A < 23.29$, and it follows that we have the reduced

bound

$$A \leq 23 \ .$$

We make a second reduction step. In stead of 10^{64} we now take $C_0 = 10^8$. We use the same notation as above. We give in the Appendix the reduced bases of the lattices Γ_i . From it we find that the shortest nonzero lattice vectors $\underline{b}_{1,i}$ all satisfy

$$|\underline{b}_{1,i}| > 1.454\times10^4 \ .$$

We computed the vectors $\underline{s}_{\ell,i}$ (see the Appendix), and found as a lower bound for the distance of their coordinates to the nearest integer, the value 0.00805. Hence *de W*, Lemma 3.5 yields

$$\min_{\underline{x}\in\Gamma_i} | \underline{x} - \underline{y}_{\ell,i} | > 2^{-1/2}\cdot 0.00805\cdot 1.454\times10^4 > 82.76 \ .$$

Reasoning as above, we find

$$|\lambda_{\ell,i}| > \left(82.76^2 - 23^2\right)^{1/2} > 79.49 \ ,$$

and

$$| \lambda_{\ell,i} - 10^8\cdot\Lambda_{\ell,i} | \leq 1 + 2\cdot A \leq 47 \ ,$$

and hence

$$|\Lambda_{\ell,i}| > 3.249\times10^{-7} \ .$$

Now (9) yields $A < 6.897$, hence we find as a new reduced bound

$$A \leq 6 \ .$$

5. COMPLETING THE PROOF

We checked (9) directly for all $6\cdot13^2 = 1014$ remaining possibilities (6 combinations of ℓ and i , and a and b running from −6 to 6). We found that no solutions with $A \geq 4$ exist, and all 150 possibilities with $A \leq 2$ are solutions of (9). With $A = 3$ there are 6 solutions of (9). We checked all 156 solutions of (9) for (6) also. It appeared that only three survived this test: with $\ell = 2$, $a = b = 0$, $i = 1, 2, 3$. These are three 'conjugated' solutions, corresponding to

$$X - Y\cdot\vartheta^{(i)} = \psi_2^{(i)} \quad \text{for} \quad i = 1, 2, 3 \ ,$$

hence to $X = -7$, $Y = 3$. Thus this is the only solution of equation (2).

This completes the proof of Antoniadis' Conjecture on equation (1).

The computations were performed on the IBM 3083 computer of the University of Leiden, and took less than 4 seconds.

APPENDIX

The values of $\vartheta^{(i)}$ for $i = 1, 2, 3$ to 75 decimal places:

$\vartheta^{(1)} =$ −2. 52891 79572 94361 73372 64844 38865 53257 69735
49113 59246 53685 00933 11868 10223 90756....

$\vartheta^{(2)} =$ 0. 16744 91911 08535 15627 44105 99528 43297 85089
86085 27948 83062 07715 60581 87728 49272....

$\vartheta^{(3)} =$ 2. 36146 87661 85826 57745 20738 39337 09959 84645
63028 31297 70622 93217 51286 22495 41483....

The values of $\log|\mu_{h,i}|$ for $h = 1, 2$, $i = 1, 2, 3$ to 65 decimal places:

$\log|\mu_{1,1}| =$ −2. 64635 90944 04415 62204 55390 06582 18159 26077
78380 15356 35680 49943 56138....

$\log|\mu_{1,2}| =$ −0. 06850 77423 86218 02662 99686 62994 38201 79350
33403 89661 00010 49667 91314....

$\log|\mu_{1,3}| =$ 2. 71486 68367 90633 64867 55076 69576 56361 05428
11784 05017 35690 99611 47452....

$\log|\mu_{2,1}| =$ −2. 22350 64735 35357 12700 57608 87706 40480 19209
27263 75564 96644 19231 53092....

$\log|\mu_{2,2}| =$ 7. 65269 58158 27467 60637 38606 30930 52800 33798
11234 52534 95207 79728 65409....

$\log|\mu_{2,3}| =$ −5. 42918 93422 92110 47936 80997 43224 12320 14588
83970 76969 98563 60497 12317....

The values of $\log|\delta_{\ell,i}|$ for $\ell = 1, 2$, $i = 1, 2, 3$ to 65 decimal places:

$\log|\delta_{1,1}| =$ −0. 89474 72881 91695 83708 76423 16001 85407 12222
60304 72711 47615 33405 82001....

$\log|\delta_{1,2}| =$ 3. 93439 06821 81768 88719 61134 86564 67126 03241
73955 28337 87981 07102 61076....

$\log|\delta_{1,3}| =$ −3. 03964 33939 90073 05010 84711 70562 81718 91019
13650 55626 40365 73696 79074....

$\log|\delta_{2,1}| =$ −0. 03448 62492 75182 03237 40083 49465 38067 86943
78373 89474 44074 60763 51767....

$\log|\delta_{2,2}| =$ 3. 38438 81770 33663 95998 90345 70330 28357 45343
27786 36047 35461 48164 53567....

$\log|\delta_{2,3}| =$ −3. 34990 19277 58481 92761 50262 20864 90289 58399
49412 46572 91386 87401 01800....

The reduced bases $(\underline{b}_{1,i}, \underline{b}_{2,i})$ of the lattices Γ_i, and the vectors $\underline{s}_{\ell,i}$ for $\ell = 1, 2$ and $i = 1, 2, 3$, with $C_0 = 10^{64}$:

$$\underline{b}_{1,1} = \begin{pmatrix} -478223012375699947666642004982086 \\ 120453456747207984447278601575495 \end{pmatrix}$$

$$\underline{b}_{2,1} = \begin{pmatrix} -165805295209341977538930622379591 \\ -473261160970439501031198835944 74 \end{pmatrix}$$

$$\underline{s}_{1,1} = \begin{pmatrix} 667206684676302834147400919977 08.4425596026.... \\ -192438721706837093057822431750 66.9785202537.... \end{pmatrix}$$

$$\underline{s}_{2,1} = \begin{pmatrix} 2\ 571615063770394620899432459411.4029118359.... \\ -741716662857046578127155251645.2390706466.... \end{pmatrix}$$

So $|\underline{b}_{1,1}| > 1.295\times10^{32}$, and the least distance to the nearest integer of the coordinates of $\underline{s}_{1,1}$ and $\underline{s}_{2,1}$ is 0.02147.... .

$$\underline{b}_{1,2} = \begin{pmatrix} 209846738140685643311301125918206 \\ -140257531056831858189008066451486 \end{pmatrix}$$

$$\underline{b}_{2,2} = \begin{pmatrix} -194152876340341550054337467123873 \\ -234912181680647399054822955079977 \end{pmatrix}$$

$$\underline{s}_{1,2} = \begin{pmatrix} 998175395933508990449912856380 18.4089281600.... \\ 107886040566173869862653140029660.8520812518.... \end{pmatrix}$$

$$\underline{s}_{2,2} = \begin{pmatrix} 858636897424207240597138486309 01.5129405558.... \\ 928042661885971227628943788200 80.6164380428.... \end{pmatrix}$$

So $|\underline{b}_{1,2}| > 2.524\times10^{32}$, and the least distance to the nearest integer of the coordinates of $\underline{s}_{1,2}$ and $\underline{s}_{2,2}$ is 0.14791.... .

$$\underline{b}_{1,3} = \begin{pmatrix} 654849759532241973159148410643 57 \\ 442701697141835438700160576611 19 \end{pmatrix}$$

$$\underline{b}_{2,3} = \begin{pmatrix} -516492677609308868582635043174615 \\ 479906642272754787848303899199678 \end{pmatrix}$$

$$\underline{s}_{1,3} = \begin{pmatrix} 289169055739097283977627599000179.5509538485.... \\ 366631115646043790533682117057 95.8407956084.... \end{pmatrix}$$

$$\underline{s}_{2,3} = \begin{pmatrix} 318684744132740681332347366265805.2237662557.... \\ 404053410839987513718503929089 89.0820715015.... \end{pmatrix}$$

So $|\underline{b}_{1,3}| > 7.904\times10^{31}$, and the least distance to the nearest integer of the coordinates of $\underline{s}_{1,3}$ and $\underline{s}_{2,3}$ is 0.08207.... .

The reduced bases $(\underline{b}_{1,i}, \underline{b}_{2,i})$ of the lattices Γ_i , and the vectors $\underline{s}_{t,i}$ for $t = 1, 2$ and $i = 1, 2, 3$, with $C_0 = 10^8$:

$$\underline{b}_{1,1} = \begin{pmatrix} -2992 \\ -14239 \end{pmatrix} \qquad \underline{b}_{2,1} = \begin{pmatrix} 14329 \\ -6123 \end{pmatrix}$$

$$\underline{s}_{1,1} = \begin{pmatrix} -5766.0429799036.... \\ -1203.9919461142.... \end{pmatrix}$$

$$\underline{s}_{2,1} = \begin{pmatrix} -222.2406238729.... \\ -46.4054676968.... \end{pmatrix}$$

So $|\underline{b}_{1,1}| > 1.454\times10^{4}$, and the least distance to the nearest integer of the coordinates of $\underline{s}_{1,1}$ and $\underline{s}_{2,1}$ is 0.00805.... .

$$\underline{b}_{1,2} = \begin{pmatrix} 20107 \\ 11762 \end{pmatrix} \qquad \underline{b}_{2,2} = \begin{pmatrix} 22006 \\ -25187 \end{pmatrix}$$

$$\underline{s}_{1,2} = \begin{pmatrix} -11313.6865015626.... \\ 10337.3759196091.... \end{pmatrix}$$

$$\underline{s}_{2,2} = \begin{pmatrix} -9732.1059161620.... \\ 8892.2772723925.... \end{pmatrix}$$

So $|\underline{b}_{1,2}| > 2.329\times10^{4}$, and the least distance to the nearest integer of the coordinates of $\underline{s}_{1,2}$ and $\underline{s}_{2,2}$ is 0.10591.... .

$$\underline{b}_{1,3} = \begin{pmatrix} -9975 \\ -20133 \end{pmatrix} \qquad \underline{b}_{2,3} = \begin{pmatrix} -19948 \\ 14166 \end{pmatrix}$$

$$\underline{s}_{1,3} = \begin{pmatrix} -11168.2983592012.... \\ 5584.7090501821.... \end{pmatrix}$$

$$\underline{s}_{2,3} = \begin{pmatrix} -12308.2544081460.... \\ 6154.7442210375.... \end{pmatrix}$$

So $|\underline{b}_{1,3}| > 2.246\times10^{4}$, and the least distance to the nearest integer of the coordinates of $\underline{s}_{1,3}$ and $\underline{s}_{2,3}$ is 0.25440.... .

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